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TRANSIENT STRESSES INDUCED BY HEATING A PLANE BOUNDARY†

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Abstract—A closed-form solution to the plane strain problem of a semi-infinite isotropic elastic medium subjected to an instantaneous line heat source at the boundary is obtained in terms of tabulated functions. This solution is regarded as fundamental and is employed to derive solutions to transient thermal stress fields for several cases of practical interest.

INTRODUCTION

AVAILABLE treatments of transient thermal stresses produced by the heating of a boundary in general have led to solutions in the form of unevaluated infinite integrals. A notable exception is the paper by Jaunzemis and Sternberg in which the problem of sudden heating of a portion of a plane boundary is considered and a series solution together with numerical results are given [1].

In the present work, to begin with, the solution to the problem of plane state of strain in a semi-infinite isotropic elastic medium subjected to instantaneous line heat source at the boundary is derived in closed form. This solution is regarded as fundamental and is employed to obtain the solution to the following cases:

1. Distribution of instantaneous heat sources on a portion of boundary.
2. Step line heat source at the boundary.
3. Step line heat doublet at the boundary.

Subsequently the case treated by Jaunzemis and Sternberg [1] is considered and a closed form solution containing error functions and their integrals is obtained.

This study is based on uncoupled quasi-static linear, plane strain, theory of thermoelasticity [2]. However, all the given solutions could be regarded as those for a semi-infinite slab with insulated faces and in a state of plane stress.

FORMULATION OF PROBLEMS

The domain of solutions is assumed to be a semi-infinite elastic isotropic homogeneous medium, shown in Fig. 1. The points are referred to either rectangular coordinates (x, y, z) or cylindrical coordinates (r, θ, z). The constants E_1, v_1, α_1, K and k designate the Young's modulus, Poisson's ratio, coefficient of linear thermal expansion, thermal conductivity and thermal diffusivity for the medium, respectively.

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$r = (x^2+y^2)^{1/2}$

$\theta = \tan^{-1} \frac{y}{x}$

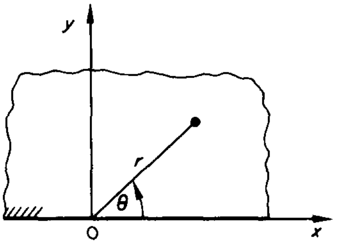

FIG. 1.

The medium is free of body and surface loads. The plane deformation of the medium is caused by a heating of a portion of the plane boundary, $y = 0$.

The stress components σ_x, σ_y and τ_{xy} are decomposed into a particular solution $(\sigma_x^p, \sigma_y^p, \tau_{xy}^p)$ and a complementary solution $(\sigma_x^c, \sigma_y^c, \sigma_{xy}^c)$ as:

$$\sigma_x = \sigma_x^p + \sigma_x^c \tag{1a}$$

$$\sigma_y = \sigma_y^p + \sigma_y^c \tag{1b}$$

$$\tau_{xy} = \tau_{xy}^p + \tau_{xy}^c. \tag{1c}$$

Further σ_x^p, σ_y^p and τ_{xy}^p, are derived from an Airy function Φ^p according to:

$$(\sigma_x^p, \sigma_y^p, \tau_{xy}^p) = \left(\frac{\partial^2\Phi^p}{\partial y^2}, \frac{\partial^2\Phi^p}{\partial x^2}, -\frac{\partial^2\Phi^p}{\partial x \partial y}\right). \tag{2}$$

The function Φ^p is a particular solution of

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \Phi^p = -E\alpha\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) T \tag{3}$$

where

$$E = E_1/(1-\nu_1^2), \qquad \alpha = \alpha_1(1+\nu_1),$$

and T is the temperature of the medium. The components σ_x^c, σ_y^c and τ_{xy}^c are obtained from suitable functions of complex variable $V(\omega)$ and $W(\omega)$ as follows [3]†:

$$\sigma_x^c = \mathrm{Re}\, V(\omega) - 2\,\mathrm{Re}\, W(\omega) - \eta\, \mathrm{Im}[V'(\omega) - W'(\omega)], \tag{4a}$$

$$\sigma_y^c = \mathrm{Re}\, V(\omega) + \eta\, \mathrm{Im}[V'(\omega) - W'(\omega)], \tag{4b}$$

$$\tau_{xy}^c = \mathrm{Im}\, W(\omega) - \eta\, \mathrm{Re}[V'(\omega) - W'(\omega)], \tag{4c}$$

where Re and Im stand for the real and the imaginary parts of the complex functions, t denotes time,

$$\omega = (\xi + i\eta),$$

$$\xi = x/(4kt)^{\frac{1}{2}},$$

$$\eta = y/(4kt)^{\frac{1}{2}},$$

$$V'(\omega) = \mathrm{d}V/\mathrm{d}\omega,$$

$$W'(\omega) = \mathrm{d}W/\mathrm{d}\omega,$$

† The formulation given here differs slightly from that of Ref. [3]. The modification is desirable for the particular cases studied in this paper.

and $i = \sqrt{-1}$. The parameter $(4kt)^{-\frac{1}{2}}$ is included in the variables for convenience. The functions $V(\omega)$ and $W(\omega)$ are determined such that the solution $(\sigma_x, \sigma_y, \tau_{xy})$ satisfies the conditions

$$\sigma_y|_{y=0} = 0, \tag{5a}$$

$$\tau_{xy}|_{y=0} = 0, \tag{5b}$$

$$(\sigma_x, \sigma_y, \tau_{xy})_{r=\infty} < \infty. \tag{5c}$$

Throughout this study it is assumed that the interior of the medium is free of heat sources. Consequently the temperature T is a solution of

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{k}\frac{\partial}{\partial t}\right)T = 0 \tag{6}$$

satisfying appropriate thermal conditions on the boundaries. The plane boundary of the medium, $y = 0$, will be assumed to be either insulated or at specified temperature except along lines or regions of source distribution.

In the subsequent sections the stresses are given in terms of real and imaginary parts of certain functions. To keep the emphasis on the essentials the discussion of the nature of the real and the imaginary parts are left for the Appendix at the end of the paper.

The corresponding solutions for the plane state of stress in slabs with insulated faces may be obtained, from the solutions given in the sequel, by replacing E and α by E_1 and α_1 respectively.

INSTANTANEOUS LINE SOURCE

Let T be the temperature field of an instantaneous line source along the z axis and lying in the otherwise insulated boundary plane $y = 0$. The temperature field for this case is well-known [4], namely

$$T = \frac{Q_0}{4\pi Kt}\,\mathrm{e}^{-\rho^2}, \tag{7a}$$

where

$$\rho = r/(4kt)^{\frac{1}{2}}, \tag{7b}$$

and Q_0 is the total heat liberated per unit length of the line source. The stress field corresponding to an instantaneous line heat source in unbounded medium is adopted as the particular solution. This stress field is given by [5]:

$$\sigma_x^p = \frac{E\alpha Q_0}{8\pi Kt}[(\cos 2\theta - 2\eta^2)\,\mathrm{e}^{-\rho^2} - \cos 2\theta]\left(\frac{1}{\rho}\right)^2 \tag{8a}$$

$$\sigma_y^p = \frac{E\alpha Q_0}{8\pi Kt}[(-\cos 2\theta - 2\xi^2)\,\mathrm{e}^{-\rho^2} + \cos 2\theta]\left(\frac{1}{\rho}\right)^2 \tag{8b}$$

$$\tau_{xy}^p = \frac{E\alpha Q_0}{8\pi Kt}[(1+\rho^2)\,\mathrm{e}^{-\rho^2} - 1]\left(\frac{1}{\rho}\right)^2 \sin 2\theta. \tag{8c}$$

On the plane boundary,

$$\sigma_y^p|_{y=0} = -\frac{E\alpha Q_0}{8\pi Kt}[(2+\xi^{-2})\,e^{-\xi^2}-\xi^{-2}],$$

$$\tau_{xy}^p|_{y=0} = 0.$$

Hence the complementary solution must be bounded at infinity and meet the conditions

$$\sigma_y^c|_{y=0} = -\sigma_y^p|_{y=0}, \tag{9a}$$

$$\tau_{xy}^c|_{y=0} = 0. \tag{9b}$$

It follows that the appropriate complex functions are $W(\omega) = 0$, and

$$V(\omega) = \frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{g(\lambda)}{\lambda-\omega}\,d\lambda, \tag{10}$$

where

$$g(\lambda) = \frac{E\alpha Q_0}{8\pi Kt}[(\lambda^{-2}+2)\,e^{-\lambda^2}-\lambda^{-2}].$$

The integral in (10), can be reduced to Laplace transform integrals by suitable transformations, leading to

$$V(\omega) = \frac{E\alpha Q_0}{8\pi Kt}\left[(2+\omega^{-2})\,e^{-\omega^2}\,\mathrm{erfc}(-i\omega)-\left(\frac{1}{\omega}\right)^2-\frac{2i}{\omega\sqrt{\pi}}\right], \tag{11}$$

where

$$\mathrm{erfc}(-i\omega) = \frac{2}{\sqrt{\pi}}\int_{-i\omega}^{\infty} e^{-\lambda^2}\,d\lambda,$$

is the complementary error function [6].† The function

$$e^{-\omega^2}\,\mathrm{erfc}(-i\omega)$$

has been extensively studied. The expansions and the real and the imaginary parts of it is given in the Appendix. Tables of this function are available when ω is expressed either in rectangular or polar coordinates [7, 8]. These functions are also discussed in Ref. [4].

For large values of ω, $V(\omega)$ behaves as

$$V(\omega) = \frac{E\alpha Q_0}{8\pi Kt}\left[-\left(\frac{1}{\omega}\right)^2+\frac{2i}{\sqrt{\pi}}\left(\frac{1}{\omega}\right)^3+0(1/\omega^4)\right], \tag{12}$$

i.e. $V(\omega)$ vanishes as $\omega \to \infty$. On the plane boundary,

$$\sigma_x^c|_{y=0} = \mathrm{Re}\,V(\xi), \qquad \sigma_y^c|_{y=0} = \mathrm{Re}\,V(\xi), \qquad \tau_{xy}^c|_{y=0} = 0.$$

However,

$$\mathrm{Re}\,V(\xi) = \frac{E\alpha Q_0}{8\pi Kt}[(2+\xi^{-2})\,e^{-\xi^2}-\xi^{-2}],$$

† The factor $(2/\sqrt{\pi})$ is not included in the definition adopted in this reference.

because

$$\operatorname{Re}\operatorname{erfc}(-i\xi) = 1.$$

Therefore, it follows that the conditions (5) are met by the complete solution.

Along the y axis σ_x^c and σ_y^c can be expressed in terms of real functions; τ_{xy} is zero along this axis.

$$\sigma_x^c|_{x=0} = \frac{E\alpha Q_0}{8\pi Kt}\left[(\eta^2+\eta^{-2})\,e^{\eta^2}\operatorname{erfc}(\eta)-\eta^{-2}+\frac{2}{\sqrt{\pi}}\frac{1}{\eta}-\frac{4}{\sqrt{\pi}}\eta\right]. \tag{13a}$$

$$\sigma_y^c|_{x=0} = \frac{E\alpha Q_0}{8\pi Kt}\left[(4-3\eta^{-2}-4\eta^2)\,e^{\eta^2}\operatorname{erfc}(\eta)+3\eta^{-2}-\frac{6}{\sqrt{\pi}}\frac{1}{\eta}+\frac{4}{\sqrt{\pi}}\eta\right]. \tag{13b}$$

The behavior of stresses for large and small values of ρ is of particular interest. For large values of this parameter the complete solution behaves as

$$\sigma_x = -\frac{E\alpha Q_0 k}{2\pi K r^2}[\cos 2\theta+\cos 4\theta+0(\rho^{-1})], \tag{14a}$$

$$\sigma_y = -\frac{E\alpha Q_0 k}{2\pi K r^2}[\cos 2\theta-\cos 4\theta+0(\rho^{-1})], \tag{14b}$$

$$\tau_{xy} = -\frac{E\alpha Q_0 k}{2\pi K r^2}[\sin 4\theta+0(\rho^{-1})]. \tag{14c}$$

It is seen that the stresses approach non-vanishing limits, as $t \to 0^+$. The limits correspond to the stresses due to a line double force on the plane boundary of the medium.

For small values of ρ the complete solution behaves as

$$\sigma_x = \frac{E\alpha Q_0}{8\pi Kt}\left[-\frac{8y}{3\sqrt{(\pi kt)}}+0(\rho^2)\right], \tag{15a}$$

$$\sigma_y = \frac{E\alpha Q_0}{8\pi Kt}\left[-\frac{y^2}{2kt}+0(\rho^3)\right], \tag{15b}$$

$$\tau_{xy} = \frac{E\alpha Q_0}{8\pi Kt}\left[\frac{xy}{2kt}+0(\rho^3)\right], \tag{15c}$$

i.e. as $t \to \infty$, the stresses approach zero as $t^{-\frac{3}{2}}$.

RECTANGULAR DISTRIBUTION OF INSTANTANEOUS SOURCES

For a uniform distribution of instantaneous sources, placed at $t = 0$ on $(-b < x < +b, y = 0)$, the stresses are derived through appropriate integration of the solution obtained in the last section. The particular solution, in the case, is

$$\sigma_x^p = \frac{E\alpha Q_1}{4\pi K}\sqrt{\frac{k}{t}}\left\{\frac{\cos\theta_1}{\rho_1}(e^{-\rho_1^2}-1)-\frac{\cos\theta_2}{\rho_2}(e^{-\rho_2^2}-1)+2[\operatorname{erf}(\xi_1)-\operatorname{erf}(\xi_2)]\,e^{-\eta^2}\right\}, \tag{16a}$$

$$\sigma_y^p = -\frac{E\alpha Q_1}{4\pi K}\sqrt{\frac{k}{t}}\left[\frac{\cos\theta_1}{\rho_1}(e^{-\rho_1^2}-1)-\frac{\cos\theta_2}{\rho_2}(e^{-\rho_2^2}-1)\right], \tag{16b}$$

$$\tau_{xy}^p = \frac{E\alpha Q_1}{4\pi K}\sqrt{\frac{k}{t}}\left[\frac{\sin\theta_1}{\rho_1}(e^{-\rho_1^2}-1)-\frac{\sin\theta_2}{\rho_2}(e^{-\rho_2^2}-1)\right], \tag{16c}$$

where Q_1 is the total heat liberated/unit area of the distribution,

$$\xi_1 = (x-b)/(4kt)^{\frac{1}{2}}, \qquad \xi_2 = (x+b)/(4kt)^{\frac{1}{2}},$$
$$\rho_1 = (\xi_1^2+\eta^2)^{\frac{1}{2}}, \qquad \rho_2 = (\xi_2^2+\eta^2)^{\frac{1}{2}},$$
$$\theta_1 = \tan^{-1}\left(\frac{y}{x-b}\right), \qquad \theta_2 = \tan^{-1}\left(\frac{y}{x+b}\right),$$

and

$$\operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\lambda^2}\,d\lambda,$$

is the error function [6].†

The appropriate complementary solution is obtained from functions $W(\omega) = 0$, and

$$V(\omega) = -\frac{E\alpha Q_1}{4\pi K}\sqrt{\frac{k}{t}}\left[\frac{1}{\omega_1}-\frac{1}{\omega_1}e^{-\omega_1^2}\operatorname{erfc}(-i\omega_1)-\frac{1}{\omega_2}+\frac{1}{\omega_2}e^{-\omega_2^2}\operatorname{erfc}(-i\omega_2)\right], \tag{17}$$

according to (4), where

$$\omega_1 = (\xi_1+i\eta),$$
$$\omega_2 = (\xi_2+i\eta).$$

Along the plane boundary $\sigma_y = 0$, $\tau_{xy} = 0$ and

$$\sigma_x|_{y=0} = \frac{E\alpha Q_1}{2\pi K}\sqrt{\frac{k}{t}}[\operatorname{erf}(\xi_1)-\operatorname{erf}(\xi_2)+\xi_1(e^{-\xi_1^2}-1)-\xi_2(e^{-\xi_2^2}-1)]. \tag{18}$$

For small values of $(1/\rho)$ the behaviour of the solution is as

$$\sigma_x = -\frac{E\alpha k Q_1}{4\pi K}\left[\frac{\cos 3\theta_1+3\cos\theta_1}{r_1}-\frac{\cos 3\theta_2+3\cos\theta_2}{r_2}+\frac{1}{\sqrt{(kt)}}O(\rho^{-3})\right], \tag{19a}$$

$$\sigma_y = \frac{E\alpha k Q_1}{4\pi K}\left[\frac{\cos 3\theta_1-\cos\theta_1}{r_1}-\frac{\cos 3\theta_2-\cos\theta_2}{r_2}+\frac{1}{\sqrt{(kt)}}O(\rho^{-3})\right], \tag{19b}$$

$$\tau_{xy} = -\frac{E\alpha k Q_1}{4\pi K}\left[\frac{\sin 3\theta_1+\sin\theta_1}{r_1}-\frac{\sin 3\theta_2+\sin\theta_2}{r_2}+\frac{1}{\sqrt{(kt)}}O(\rho^{-3})\right]. \tag{19c}$$

† The factor $(2/\sqrt{\pi})$ is not included in the definition adopted in this reference.

Clearly the stresses approach a well-defined limit as t approaches zero through positive values. The limit is the stress field due to two line forces of equal magnitude and opposite sign acting in the positive and the negative x-directions along the lines ($x = b, y = 0$) and ($x = -b, y = 0$), respectively. The magnitude of each force/unit length of the line is $(E\alpha kQ_1/2K)$ [9].

STEP LINE SOURCE

The stresses due to a line source of step time dependency is derived through integration of the instantaneous line source solution with respect to time. For this case the particular solution is

$$\sigma_x^p = \frac{E\alpha Q_2}{8\pi K}\left[\frac{\cos 2\theta}{\rho^2}(e^{-\rho^2}-1)-E_1(\rho^2)\right], \tag{20a}$$

$$\sigma_y^p = \frac{E\alpha Q_2}{8\pi K}\left[\frac{\cos 2\theta}{\rho^2}(1-e^{-\rho^2})-E_1(\rho^2)\right], \tag{20b}$$

$$\tau_{xy}^p = \frac{E\alpha Q_2}{8\pi K}\frac{\sin 2\theta}{\rho^2}(e^{-\rho^2}-1), \tag{20c}$$

where

$$E_1(\rho^2) = \int_{\rho^2}^{\infty} e^{-\lambda}\frac{d\lambda}{\lambda},$$

is the exponential integral [6] and Q_2 is the amount of heat liberated per unit length of the line source/unit time.

The complex functions for the determination of complementary solution are $W(\omega) = 0$ and

$$V(\omega) = \frac{E\alpha Q_2}{8\pi K}\left[\frac{1}{\omega^2}e^{-\omega^2}\operatorname{erfc}(-i\omega)-\left(\frac{1}{\omega}\right)^2-\frac{2i}{\sqrt{\pi}}\frac{1}{\omega}-2\int_{\infty}^{\omega}e^{-\lambda^2}\operatorname{erfc}(-i\lambda)\frac{d\lambda}{\lambda}\right]. \tag{21}$$

For small values of ω the function $V(\omega)$ behaves as

$$V(\omega) = \frac{E\alpha Q_2}{8\pi K}\left[-2\log\omega-\gamma-1-\frac{16i}{3\sqrt{\pi}}+\frac{3}{2}\omega^2+\frac{64i}{45\sqrt{\pi}}\omega^3+0(\omega^4)\right], \tag{22}$$

where γ is the Euler's constant [6]. For large values of ω, on the other hand, the behaviour of $V(\omega)$ is as

$$V(\omega) = \frac{E\alpha Q_2}{8\pi K}[-\omega^{-2}+0(\omega^{-3})]. \tag{23}$$

The state of stress is regular at the source location for $t > 0^+$. This is in contrast to the case of unbounded medium in which stresses exhibit logarithmic singularity as $\rho \to 0$. In fact, in semi-infinite medium the stresses approach to a uniform compression in the x-direction, as $\rho \to 0$, according to:

$$\sigma_x = \frac{E\alpha Q_2}{8\pi K}\left[-2+\frac{16y}{3\sqrt{(\pi kt)}}+0(\rho^2)\right] \tag{24a}$$

$$\sigma_y = \frac{E\alpha Q_2}{8\pi K}\left[\frac{y^2}{2kt}+0(\rho^3)\right] \tag{24b}$$

$$\tau_{xy} = \frac{E\alpha Q_2}{8\pi K}\left[-\frac{xy}{2kt}+0(\rho^3)\right]. \tag{24c}$$

The singular stress field in an unbounded medium, due to a steady line heat source, develops because the thermal expansion of the material at and around the source location is constrained by the rest of the medium. This constraining effect is not present when the source is at the boundary.

The uniform compression in the x direction, indicated by the solution, in the limit as $t \to \infty$, could be explained by noting that strictly speaking the semi-infinite solid is not free to expand in the x direction. Also according to (21) the stresses approach the limit at different rates.

It is evident that conditions stronger than (5c), i.e. vanishing stresses at points far from the heated region, is not appropriate for all times. In the next section the case of line heat doublet is treated for which, in contrast to the case of line heat source, the total heat supply to the medium is zero and hence a well defined steady state temperature function exists. It is shown that, as $t \to \infty$, all stress components approach zero in that case.

Components of stress assume simple forms along coordinate planes. Along the free boundary, for instance, $\sigma_y = 0$, $\tau_{xy} = 0$ and

$$\sigma_x|_{y=0} = \frac{E\alpha Q_2}{4\pi K}\frac{1}{\xi^2}(e^{-\xi^2}-1). \tag{25}$$

STEP LINE DOUBLET PARALLEL TO THE BOUNDARY

The stresses associated with a step line doublet, the axis of which coincide with the x axis, are obtained by differentiating the step line-source solution, derived in the preceding section. For this case the particular solution is

$$\Phi^p = \frac{E\alpha Q_3}{8\pi K}\left[\frac{x}{\rho^2}(e^{-\rho^2}-1)-xE_1(\rho^2)\right], \tag{26}$$

where Q_3 is the doublet strength/unit length of the line per unit time. The complementary solution, on the other hand, is derived from complex functions $W(\omega) = 0$ and

$$V(\omega) = \frac{E\alpha Q_3}{8\pi K}\frac{1}{\sqrt{kt}}\left[\frac{1}{\omega^3}+\frac{2i}{\sqrt{\pi}}\frac{1}{\omega^2}-\left(\frac{2}{\omega}+\frac{1}{\omega^3}\right)e^{-\omega^2}\operatorname{erfc}(-i\omega)\right]. \tag{27}$$

For small values of $(1/\rho)$, the stresses behave as

$$\sigma_x = \frac{E\alpha Q_3}{8\pi Kr}\left[\frac{\cos 3\theta - 3\cos 5\theta}{\rho^2} + 0(\rho^{-3})\right], \tag{28a}$$

$$\sigma_y = \frac{E\alpha Q_3}{8\pi Kr}\left[\frac{3\cos 3\theta - 3\cos 5\theta}{\rho^2} + 0(\rho^{-3})\right], \tag{28b}$$

$$\tau_{xy} = \frac{E\alpha Q_3}{8\pi Kr}\left[\frac{-5\sin 3\theta + 3\sin 5\theta}{\rho^2} + 0(\rho^{-3})\right]. \tag{28c}$$

For large values of $(1/\rho)$, on the other hand, the stresses behave like

$$\sigma_x = \frac{E\alpha Q_3}{8\pi K}\frac{1}{\sqrt{(kt)}}[\xi + 0(\rho^2)], \tag{29a}$$

$$\sigma_y = \frac{E\alpha Q_3}{8\pi K}\frac{1}{\sqrt{(kt)}}[0(\rho^2)], \tag{29b}$$

$$\tau_{xy} = \frac{E\alpha Q_3}{8\pi K}\frac{1}{\sqrt{(kt)}}[-2\eta + 0(\rho^2)], \tag{29c}$$

i.e. stresses approach zero, as $t \to \infty$. Once again the stresses approach zero at different rates.

A line doublet simultaneously heats one half and cools the other half of the medium. Also in the limit as $t \to \infty$, the temperature function due to step line doublet approaches a well-defined harmonic function decaying as $(1/r)$ in all directions. Consequently any constraining at $x = \pm\infty$ would not result in uniform compression in the x-direction, as in the case of step line heat source discussed in the preceding section.

DISCONTINUOUS TEMPERATURE RISE†

In the hope of constructing a more compact solution through use of complex functions the problem treated by Jaunzemis and Sternberg [1] is considered here again.

Initially the medium is at zero temperature everywhere. At $t = 0^+$, the temperature of the strip $(-b < x < +b, y = 0)$ is raised to the constant value T_0. The rest of the boundary is at zero temperature at all times.

The well-known temperature function for this case is [4]

$$\pi T = T_0 \int_{\xi_1}^{\xi_2} e^{-(\lambda^2+\eta^2)} \frac{\eta \, d\lambda}{\lambda^2 + \eta^2}. \tag{30}$$

† Direct derivation of the solution is more convenient in this case. The problem is essentially identical to the case of continuous distribution of doublets on the boundary.

Using the Goodier's method [2] and the temperature function (30) the following particular solution is obtained:

$$\sigma_x^p = \frac{E\alpha T_0}{4\pi}\left[\frac{\sin 2\theta_1}{\rho_1^2}(e^{-\rho_1^2}-1)-\frac{\sin 2\theta_2}{\rho_2^2}(e^{-\rho_2^2}-1)+4\int_{\xi_2}^{\xi_1} e^{-(\lambda^2+\eta^2)}\frac{\eta\, d\lambda}{\lambda^2+\eta^2}\right] \tag{31a}$$

$$\sigma_y^p = \frac{E\alpha T_0}{4\pi}\left[\frac{\sin 2\theta_1}{\rho_1^2}(1-e^{-\rho_1^2})-\frac{\sin 2\theta_2}{\rho_2^2}(1-e^{-\rho_2^2})\right] \tag{31b}$$

$$\tau_{xy}^p = \frac{E\alpha T_0}{4\pi}\left[\frac{\cos 2\theta_1}{\rho_1^2}(1-e^{-\rho_1^2})-\frac{\cos 2\theta_2}{\rho_2^2}(1-e^{-\rho_2^2})+E_1(\rho_1^2)-E_1(\rho_2^2)\right]. \tag{31c}$$

The particular solution is zero at infinity. On the plane boundary σ_y^p is zero and

$$\tau_{xy}^p|_{y=0} = \frac{E\alpha T_0}{4\pi}[(1-e^{-\xi_1^2})\xi_1^{-2}-(1-e^{-\xi_2^2})\xi_2^{-2}+E_1(\xi_1^2)-E_1(\xi_2^2)].$$

The complementary solution must, therefore, be bounded at infinity and meet the conditions

$$\sigma_y^c|_{y=0} = 0, \qquad \tau_{xy}^c|_{y=0} = -\tau_{xy}^p|_{y=0}.$$

Using the techniques employed in previous sections it can be shown that the appropriate complementary solution is $V(\omega) = 0$ and

$$W(\omega) = \frac{E\alpha T_0}{4\pi}\left[\frac{i}{\omega_2^2}-\frac{i}{\omega_1^2}+\frac{i}{\omega_1^2}e^{-\omega_1^2}\operatorname{erfc}(-i\omega_1)-\frac{i}{\omega_2^2}e^{-\omega_2^2}\operatorname{erfc}(-i\omega_2)+\frac{2}{\sqrt{\pi}}\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right)\right.$$
$$\left.+2i\int_{\omega_2}^{\omega_1} e^{-\lambda^2}\operatorname{erfc}(-i\lambda)\frac{d\lambda}{\lambda}\right]. \tag{32}$$

The expressions (31) and (32) constitute the solution. Apart from an integral in (31a) and an integral in (32) these expressions contain only tabulated well-known functions. The integral in (32) is identical to the one in (21), i.e. in the solution for step line source at the boundary. The integral in (31a) is none other than the temperature function (30). These integrals have tabulated integrands and well-defined limits. The real and the imaginary parts of these integrals and appropriate expansions for them are given in the Appendix.

The behaviour of $W(\omega)$ for small and large values of ρ is, respectively, as follows:

$$W(\omega) = \frac{E\alpha T_0}{4\pi}\left[2i\log\frac{r_1}{r_2}-2(\theta_1-\theta_2)+\frac{8b}{3\sqrt{(\pi kt)}}+\frac{3}{2}\rho_1^2(\sin 2\theta_1-i\cos 2\theta_1)\right.$$
$$\left.-\frac{3}{2}\rho_2^2(\sin 2\theta_2-i\cos 2\theta_2)+0(\rho^3)\right] \tag{33a}$$

$$W(\omega) = -\frac{E\alpha T_0}{4\pi}\left[\frac{\sin 2\theta_1+i\cos 2\theta_1}{\rho_1^2}-\frac{\sin 2\theta_2+i\cos 2\theta_2}{\rho_2^2}+0(\rho^{-3})\right]. \tag{33b}$$

The stresses for small values of ρ behave as

$$\sigma_x = -E\alpha T_0\left[\frac{4}{3\pi}\frac{b}{\sqrt{(\pi kt)}}+0(\rho^2)\right], \tag{34a}$$

$$\sigma_y = -E\alpha T_0\left[\frac{by}{2\pi kt}+0(\rho^3)\right], \tag{34b}$$

$$\tau_{xy} = -E\alpha T_0\left[\frac{y}{3\pi\sqrt{(\pi kt)}}+0(\rho^2)\right], \tag{34c}$$

i.e. they approach zero like $(b^2/kt)^{\frac{1}{2}}$ as $t \to \infty$.

For large values of ρ, i.e. $r \gg b$ and $(kt/b^2) \to 0^+$, stresses behave as

$$\sigma_x = \frac{E\alpha T_0}{4\pi}\left[\frac{\sin 4\theta_1}{\rho_1^2}-\frac{\sin 4\theta_2}{\rho_2^2}+0(\rho^{-3})\right], \tag{35a}$$

$$\sigma_y = \frac{E\alpha T_0}{4\pi}\left[\frac{2\sin 2\theta_1-\sin 4\theta_1}{\rho_1^2}-\frac{2\sin 2\theta_2-\sin 4\theta_2}{\rho_2^2}+0(\rho^{-3})\right], \tag{35b}$$

$$\tau_{xy} = \frac{E\alpha T_0}{4\pi}\left[\frac{\cos 2\theta_1-\cos 4\theta_1}{\rho_1^2}-\frac{\cos 2\theta_2-\cos 4\theta_2}{\rho_2^2}+0(\rho^{-3})\right]. \tag{35c}$$

CONCLUSIONS

Combining particular solutions with appropriate complementary solutions several physically interesting thermal stress problems have been solved in closed form. The solutions describe states of plane strain in half-space that is heated over portion of the plane boundary (the remainder of the boundary being insulated or kept at some reference temperature). The closed form of the solutions given, permits detailed study of the behaviour of the stress fields at different times and locations. It is shown that the singular state of stress around a step line-source (or a step-line-doublet, etc.) in the interior of the medium converts to a regular state as the line source (or a doublet, etc.) is moved from the interior to a free plane boundary. Furthermore, in general, the boundedness is the strongest condition which could be imposed on stresses, at infinity, at all times. However, in the cases where the steady state temperature field is a well defined, harmonic, bounded, function the stresses vanish, at all times, at points very far from the heated region. The solutions constructed are expressed in terms of tabulated functions and certain simple integrals of these functions with definite limits. Therefore these solutions are suitable for numerical calculations.

Examination of the solutions for step line-source, step line-doublet and variable temperature rise constructed here, indicates that in general, the rate of decay of stress with distance from the heated region increases when the rate of heat input is decreased. The question of the rate of decay of stresses caused by localized heating is related to the validity of an overall type of Saint Venant principle [10]; a detailed treatment of this matter requires further investigation.

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APPENDIX

Throughout the paper the complementary stress field is expressed in terms of complex functions $V(\omega)$ and $W(\omega)$. Presently computer programs are available to calculate the real and imaginary parts of variety of functions of complex variable ω. However, for the sake of completeness and for the purpose of convenient use of the solutions the real and the imaginary parts of all the complex functions appearing in the solutions are given below in the forms suitable for numerical calculations.

The nature of the integral appearing in (30) and in (31a) is also discussed.

1. *Real and imaginary parts of algebraic functions*

$$\operatorname{Re}\omega^n = \rho^n \cos n\theta, \qquad \operatorname{Im}\omega^n = \rho^n \sin n\theta.$$

2. *Real and imaginary parts of transcendental functions*

The only transcendental function appearing in the solution is

$$f(\omega) = e^{-\omega^2}\operatorname{erfc}(-i\omega) = e^{-\omega^2} + e^{-\omega^2}\operatorname{erf}(i\omega).$$

Extensive informations are available on this function [6]. Also tables giving the real and the imaginary parts of $f(\omega)$ is in hand (see [4, 7, 8]).

To begin with

$$\operatorname{Re}(e^{-\omega^2}) = e^{-\rho^2\cos 2\theta}[\cos(\rho^2 \sin 2\theta)],$$

$$\operatorname{Im}(e^{-\omega^2}) = e^{-\rho^2\cos 2\theta}[\sin(\rho^2 \sin 2\theta)].$$

On the other hand the known expansion

$$e^{-\omega^2}\operatorname{erf}(i\omega) = \frac{2i}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-)^n\omega^{2n+1}}{(\frac{3}{2})_n},$$

is available, where

$$(\tfrac{3}{2})_n = (\tfrac{3}{2})(\tfrac{3}{2}+1)(\tfrac{3}{2}+2)\ldots(\tfrac{3}{2}+n-2)(\tfrac{3}{2}+n-1).$$

Hence

$$\mathrm{Re}[e^{-\omega^2}\,\mathrm{erf}(i\omega)] = \frac{-1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-)^n\rho^{2n+1}}{(\frac{3}{2})_n}\sin(2n+1)\theta,$$

$$\mathrm{Im}[e^{-\omega^2}\,\mathrm{erf}(i\omega)] = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-)^n\rho^{2n+1}}{(\frac{3}{2})_n}\cos(2n+1)\theta.$$

For very large values of ρ the asymptotic expansion of (ω) may be used to obtain:

$$\mathrm{Re}\,f(\omega) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{N-1}(-)^n(\tfrac{1}{2})_n\rho^{-2n-1}\sin(2n+1)\theta+0(\rho^{-2N-1}),$$

$$\mathrm{Im}\,f(\omega) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{N-1}(-)^n(\tfrac{1}{2})_n\rho^{-2n-1}\cos(2n+1)\theta+0(\rho^{-2N-1}).$$

3. *The function* $g(\omega)$

The integral

$$g(\omega)-g(\omega_0) = \int_{\omega_0}^{\omega} f(\lambda)\frac{d\lambda}{\lambda} = \int_{\omega_0}^{\omega} e^{-\lambda^2}\,\mathrm{erfc}(-i\lambda)\frac{d\lambda}{\lambda},$$

where ω_0 is a complex constant also appears in the solutions. Now

$$g(\omega) = -\tfrac{1}{2}E_1(\omega^2)+h(\omega)$$

where E_1 is the exponential integral [6] and

$$h(\omega) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-)^n\omega^{2n+1}}{(2n+1)(\frac{3}{2})_n}.$$

Hence

$$\mathrm{Re}\,h(\omega) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)}\frac{\rho^{2n+1}}{(\frac{3}{2})_n}\cos(2n+1)\theta,$$

$$\mathrm{Im}\,h(\omega) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-)^n}{(2n+1)}\frac{\rho^{2n+1}}{(\frac{3}{2})_n}\sin(2n+1)\theta.$$

On the other hand

$$\mathrm{Re}\,E_1(\omega^2) = -\gamma-\log\rho^2-\sum_{n=1}^{\infty}\frac{(-)^n\rho^{2n}}{n(n!)}\cos 2n\theta,$$

$$\mathrm{Im}\,E_1(\omega^2) = -2\theta-\sum_{n=1}^{\infty}\frac{(-)^n\rho^{2n}}{n(n!)}\sin 2n\theta,$$

where γ is the Euler's constant.

For very large values of ρ the following asymptotic expansions are available:

$$\mathrm{Re}\,g(\omega) = \frac{-1}{\sqrt{\pi}}\sum_{n=0}^{N-1}(\tfrac{1}{2})_n\rho^{-2n-1}\frac{\sin(2n+1)\theta}{(2n+1)}+0(\rho^{-2N-1}),$$

$$\mathrm{Im}\,g(\omega) = \frac{-1}{\sqrt{\pi}}\sum_{n=0}^{N-1}(\tfrac{1}{2})_n\rho^{-2n-1}\frac{\cos(2n+1)\theta}{(2n+1)}+0(\rho^{-2N-1}).$$

4. *Certain temperature function*

The function

$$R(\xi_1, \xi_2, \eta) = \int_{\xi_2}^{\xi_1} e^{-(\lambda^2+\eta^2)} \frac{\eta \, d\lambda}{\lambda^2+\eta^2}$$

appears in (30) and (31a). To find a more convenient presentation of this function which is suitable for numerical calculations, in Ref. [1] the function is first expressed in terms of iterated error integrals and then Rosser's expansion is employed to present R in the form of series of error and other incomplete gamma functions.

Calculation of R should not present major difficulties except at the vicinity of points of discontinuity, i.e. ($x = \pm b, y = 0$). However at the vicinity of the plane boundary the following convenient presentation may be employed:

$$R = \theta_2 - \theta_1 + \frac{by}{2kt} - \eta \sum_{n=1}^{\infty} \frac{(-)^n}{(2n+1)(n+1)!}(\xi_1^{2n+1} - \xi_2^{2n+1}) + 0(\eta^3).$$

For larger values of η the expansion

$$R = \theta_2 - \theta_1 + \frac{by}{2kt} - \eta \sum_{n=1}^{\infty} \frac{(-)^n}{(n+1)!}\left[\int_{\xi_2}^{\xi_1} (\lambda^2+\eta^2)\, d\lambda\right],$$

is more suitable. Using binomial expansions the last integral can be easily evaluated.

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Абстракт—Получается решение в замкнутом виде, выраженное табулированными функциями, для задачи плоской деформации, касающееся полубесконечной, изотропной, упругой среды, подверженной действию многовенного, линейного источника тепла на краю. Решение считеется фундаментальным. Оно используется с целью определения решений для нестационарных полей термических напряжении, для некоторых случаев, имеющих практическое значение.